Numerical study of scars in a chaotic billiard

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We study numerically the scaling properties of scars in stadium billiard. Using the semiclassical criterion, we have searched systematically the scars of the same type through a very wide range, from ground state to as high as the 1 millionth state. We have analyzed the integrated probability density along the periodic orbit. The numerical results confirm that the average intensity of certain types of scars is independent of $\hbar$ rather than scales with $\sqrt{\hbar}$. Our findings confirm the theoretical predictions of Robnik (1989). [S1063-651X(97)14205-2]

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Eigenstates of a bound quantum system whose classical counterpart is chaotic are of great interest in the fast developing field of ''quantum chaos.'' Among many others, scars are one of the most interesting and striking topics. Since its discovery [1,2], much progress has been achieved. On the theoretical side, Bogomolny [3] developed semiclassical theory of scars in configuration space, and Berry [4] performed a similar analysis in phase space using the Wigner function. According to this theory, the intensity of a scar goes as $\sqrt{\hbar}$. Based on the semiclassical evaluation of the Green function of the Schrödinger equation in terms of the classical orbit, Robnik [5] has developed a theory, and predicted that if the scar is supported by many periodic orbits, the maximal intensity of the scar is independent of $\hbar$, although its geometry can be determined by Bogomolny's theory. Most recently, Klakow and Smilansky [6] used a scattering quantization approach for this problem. Parallel to the theoretical developments, there have also been many numerical [7,8] and experimental studies [9].

Unfortunately, due to the limit of the numerical techniques and the computer facilities, most of the numerical studies up to now are restricted to a very low energy range, which is too low to verify the theoretical predictions in the very far semiclassical limit, particularly for Robnik's theory. In this paper, by using our numerical code of the improved plane wave decomposition method (PWDM) (for more details about Heller's PWDM, please see [10], while for the improved PWDM, we will discuss it in another paper [14]), we are successful to go as high as 1 millionth state, which is very deep in semiclassical regime for the stadium billiard. With the help of the semiclassical criterion [5], we found many consecutive scars in several different energy ranges, which spans 2 orders of magnitude in the wave vector, therefore, we are able to study the properties of scars, such as the intensity and profiles in the very far semiclassical limit.

To make the numerical data significant, we need enough ensembles of scars of the same type. Therefore, our first step is to collect scars of the same type in a wide range of energy. We begin from a very low state, e.g., the ground state. As long as we find the first scar, say, e.g., at wave vector k_0, then we can use the semiclassical criterion to estimate the next scar. According to the semiclassical theory [3–5], the scar is most likely to occur if quantized, i.e.,

$$S = 2\pi\hbar\left(n + \frac{\alpha}{2}\right), \quad n = 0,1,2, \ldots . \tag{1}$$

S is the action along the periodic orbit, α is the Maslov phase. It must be pointed out that the semiclassical theory cannot predict the individual state at which the scar will occur. Instead, if we have already found one scar, say, at k_0, then the semiclassical theory tells us that the eigenstates at the wave vector of $k_0 \pm \Delta k$ most likely will be scarred. $\Delta k = 2\pi\hbar/\mathcal{L}$, $\mathcal{L}$ is the length of the periodic orbit. In our study we put $\hbar = 1$, so the inverse of the wave vector k plays the role of $\hbar$; i.e., k goes to infinity indicates the semiclassical limit. It has been verified in our numerical study that this criterion is very helpful and very successful in searching for and collecting scars. As we shall see later, in many cases this criterion is accurate within one mean level spacing, namely, the scar occurs at the eigenstate whose eigenenergy roughly equals the predicted energy.

With the help of the semiclassical quantization criterion equation (1), we have found about 100 examples of the same type of scar at different energy ranges. One such example is shown in Fig. 1. The eigenvalue of this eigenstate is $k = 1328.153\,849$, which corresponds to the sequential number 250 034 for odd-odd parity, and to the index of about 1 001 408 when all parities are taken into account. To our surprise, in addition to this one, we have found quite a few examples of this type of scarred state in such a high level. This implies that the scars survive the semiclassical limit. Does this finding contradict Shnirlman's theorem [11], which states that as the energy goes to infinity, the probability density of most eigenstates of a chaotic billiard approaches a uniform distribution? To test this, we have investigated the statistics of the probability density distribution of the wave function, and found that it is an excellent Gaussian distribution, although there is a pronounced density around the periodic orbit.

In order to understand the scar properties quantitatively, we have investigated the following pronounced (excess) intensity in a thin tube along the periodic orbit (see Fig. 2), which is defined by

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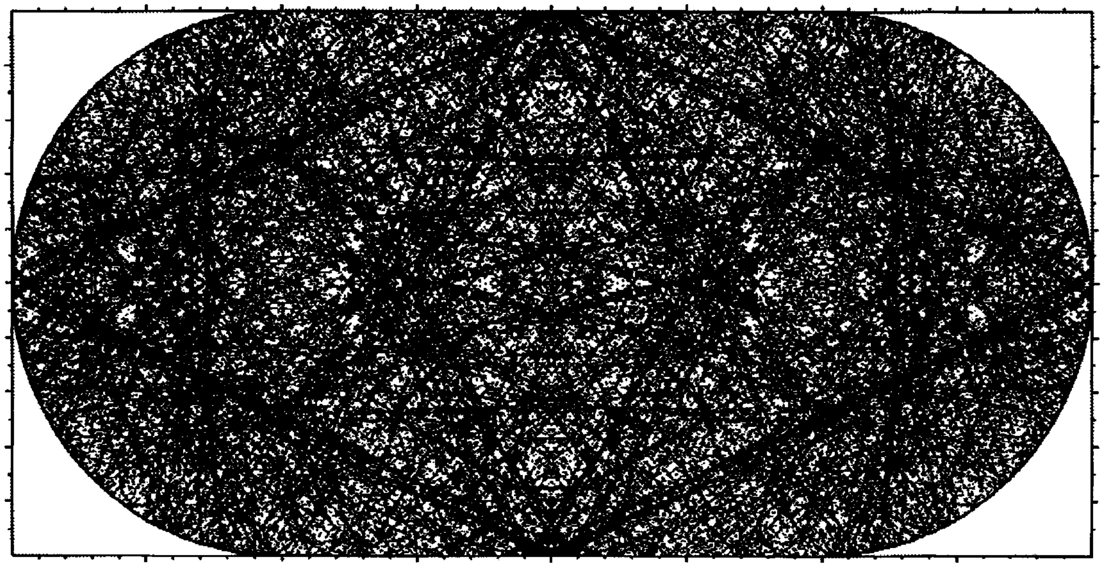

FIG. 1. The probability density plot for a scarred eigenstate of odd-odd parity. The wave number $k=1328.153\,849$, which corresponds to index 250 034 using the Weyl formula (odd-odd); thus it corresponds to approximately the 1 001 408th eigenstate for the total billiard. The scar is obviously supported by the diamond-shaped periodic orbit shown in Fig. 2. The stadium has the parameter of circle radius $R=1$ and the straight line length 2. In this figure, the unit length is about 211 de Broglie wavelengths.

$$I=\frac{\int\psi^2(\mathbf{x})d\mathbf{x}}{\int\langle\psi^2(\mathbf{x})\rangle d\mathbf{x}}-1, \tag{2}$$

where $\psi(\mathbf{x})$ is the eigenfunction at $\mathbf{x}$. $\langle\psi^2(\mathbf{x})\rangle$ is the average probability density inside the billiard, which is $1/\mathcal{A}$ according to the semiclassical theory [11–13]. $\mathcal{A}$ is area of the billiard. The integral is taken over a thin tube around the periodic orbit, which is presented in Fig. 2.

According to Robnik's theory [5], although the geometry of a scar is determined by a single short periodic orbit, the intensity profile is nevertheless determined by the sum of contributions from similar but longer periodic orbits, which ''live'' in the homoclinic neighborhood close to the stable and unstable manifolds of the primitive orbit. Taking into account all these orbits, the pronounced intensity of the scar defined by Eq. (2) can be described by the following formula:

$$I\approx\nu\sum_{n=1}^{\infty}\frac{\sin(nS_1/\hbar)}{\sinh(n\lambda\tau/2)}-1, \tag{3}$$

where S_1 is the action along the primitive periodic orbit, λ is the Lyapunov exponent of the primitive orbit with the period of τ, the summation over n is due to the repetitions of the orbit, and ν is the number of contribution orbits. Equation (3) states that *the maximal intensity* of the scar, when supported by many periodic orbits, is independent of $\hbar$. This thoeretical prediction is different from that of Bogomolny. But, it does not contradict that of Bogomolny at all; instead, it is an extension of Bogomolny's theory to the scars caused by many periodic orbits. These two theories describe different types of scars. Indeed, we have also found the scar types whose intensity depends on $\sqrt{\hbar}$, which is exactly predicted by Bogomolny's theory. However, since other authors [8] have already verified this theoretical prediction, we will not repeat this in this paper; we shall concentrate on the scars that cannot be determined by Bogomolny's theory, but can be described by Robnik's theory.

In Fig. 3, we show six representative examples of this intensity versus the width of the tube (D) in units of de Broglie wavelength around the periodic orbit. These six examples are the same type of scar, namely, the diamond-shaped scar shown in Fig. 1. They go from the very low state $k=10.241\,095$ to the very deep semiclassical regime at $k=1328.153\,849$.

The first thing one can see from these profile figures is that the scar intensity has a maximum at the width of about 1–2 de Brodglie wavelengths from the periodic orbit. This can be explained by Robnik's theory. The semiclassical waves associated with individual daughter orbits interfere constructively with each other only within a tube of width 1–2 de Broglie wavelengths. The second important result from this figure is that the magnitude of the maximum does not change too much although the eigenenergy changes more than 100 times.

Moreover, after checking the eigenenergies of these six examples carefully, we have found that the semiclassical criterion works very well, although we go from one scar state to another by jumping even up to a few hundred scarred states.

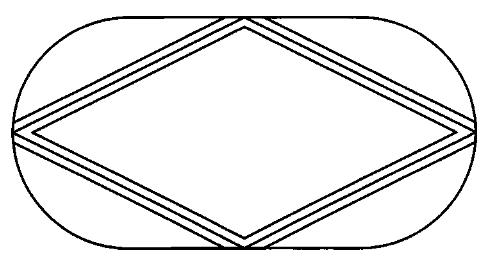

FIG. 2. The integral region around the periodic orbit that is taken in Eq. (2). The width of the tube is D measured perpendicular to the periodic orbit.

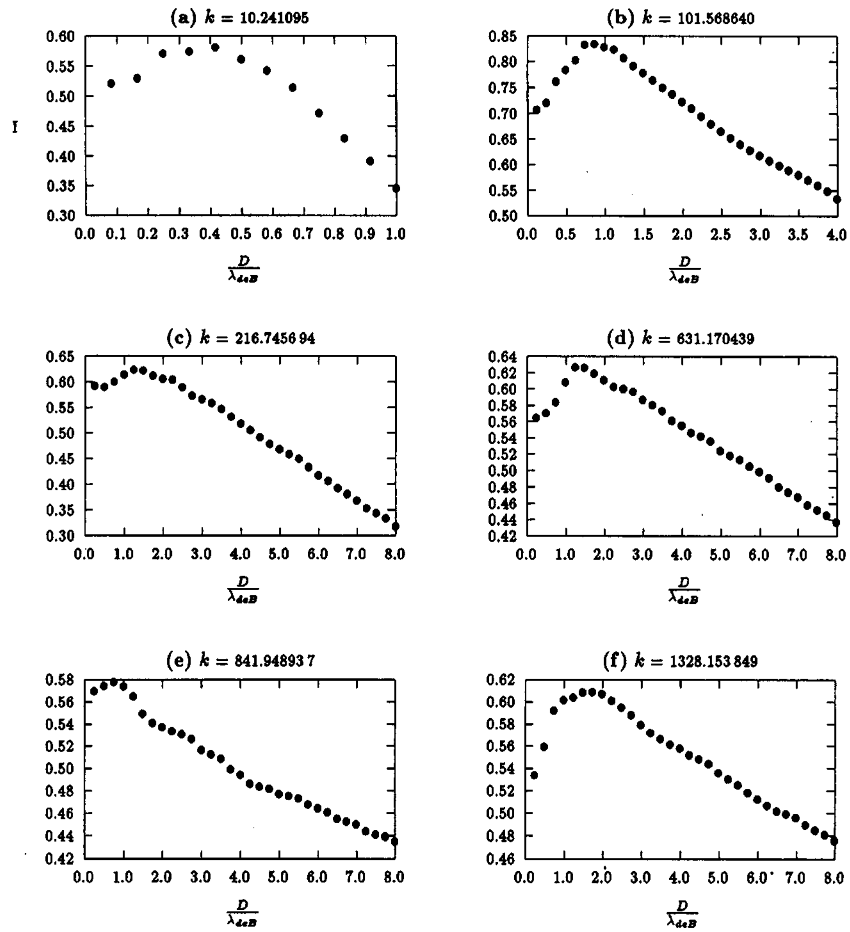

FIG. 3. The integrated scar intensity I vs the width of the integrating tube in unit of the de Broglie wavelength for the scar type shown in Fig. 1. We show six different eigenstates at different energy ranges.

For instance, starting from the first eigenvector $k_0 = 10.241\,095$, if we go through 65 scarred states, we have $k = k_0 + 65\Delta k = 101.563\,684$, which is very close to the exact one $k_{\mathrm{exact}} = 101.568\,640$. (In this paper, we study only the eigenstates with odd-odd parity, so the length of the periodic orbit shown in Fig. 2 is $\mathcal{L} = 2\sqrt{5}$, rather than $4\sqrt{5}$ for the total billiard, thus, $\Delta k = 2\pi/\mathcal{L} = 1.404\,96$). The deviation is less than one mean level spacing. This procedure applies also to many other scarred states and it can be verified readily for other states given in Fig. 3.

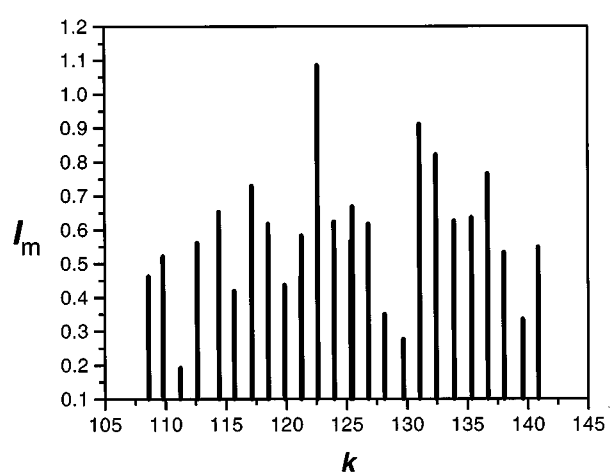

FIG. 4. The maximum of integrated scar intensity vs wave number k around $k = 125$. The type of scar is the same as shown in Fig. 1 (the diamond shape). Here we see clearly that the wave number interval between two consecutive scars is very close to $2\pi/\mathcal{L}$ (= 1.40496), as predicted by the semiclassical quantization condition Eq. (1).

The next very important question is that how does this maximal integrated intensity depend on the eigenenergy or the $\hbar$? Firstly, our numerical results show that around a certain k, it changes from scarred state to state. This is shown in

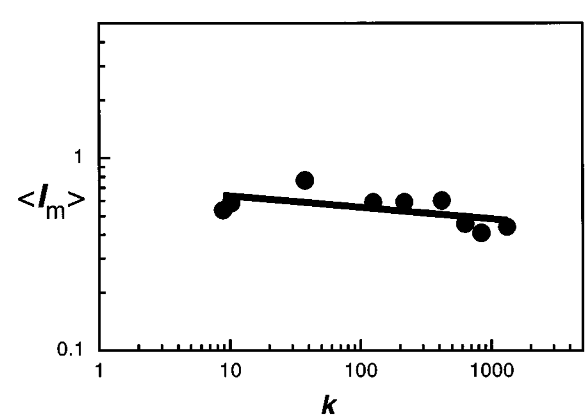

FIG. 5. The locally averaged (over a small group of consecutive scarred states) maximum of integrated scar excess intensity vs wave number k. The bullet represents the numerical data, and the solid line is the best least-squares fit.

Fig. 4, where we plot 26 consecutive scarred states around $k=125$. (Note that there are two cases in which two consecutive eigenstates are near degenerate, thus both of them are scarred.) Again, from this figure we can see clearly that the semiclassical criterion (1) works excellently. The interval between two scarred states is almost constant and approximately equals $2\pi/\mathcal{L}$. The maximal integrated intensity, however, fluctuates from state to state, which cannot be explained by the existing semiclassical approaches. This is still an open problem deserving of further theoretical and numerical investigations.

The results shown in Fig. 4 imply that in order to make the study of the dependence of the maximal integrated intensity on energy significant, we should take certain kinds of ensemble averaging. In our numerical study, we have performed such averaging around certain k over many scarred states (usually about 10 states). The averaged data are plotted in Fig. 5. The best least-squares fit gives rise to

$$\langle I_m \rangle = 0.73/k^a, \quad a = 0.06 \pm 0.03, \tag{4}$$

here $\langle\ \rangle$ means the local average. The exponent a is very close to zero and is far from 1/2 as predicted by Bogomolny's theory. This fact means that the maximal integrated intensity does not depend on the energy or $\hbar$ for the scar type shown in this paper. This discovery is very different from previous ones [8] and cannot be explained by the semiclassical theory of Bogomolny [3] and Berry [4], however, it confirms quantitatively the theoretical prediction of Robnik [5], which states that the maximal intensity of a scar is inde--pendent of $\hbar$ if the scar is supported by many orbits, as mentioned above.

In this paper, we have studied intensively the scars in a stadium billiard, and have shown numerically that the semiclassical criterion (1) works very well from very low state to that in the very far semiclassical limit. Furthermore, we have analyzed the scaling property of a scar with $\hbar$ and found that for the scar type shown in this paper, the maximal integrated density fluctuates from scarred state to state, but the local average intensity does not change with energy. This finding confirms Robnik's scar theory of multiple periodic orbits [5].

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